ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 7, January 2014

# A new Concept of Duality for Linear Fractional Programming Problems

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***Abstract; - In this paper a new concept of duality is given for the linear fractional programming (LFP) problem in which the objective function is a linear fractional function and where the constraint functions are in the form of linear inequalities. Our result is based on transforming the linear fractional programming problem to an equivalent linear programming with the same dimension. A simple example is given to clarify the developed theory in this paper.***

***Keywords:*** **linear fractional programming- Linear programming- duality concept complementary slackness**

## I. INTRODUCTION

Linear fractional maximum problems (i.e. ratio objective that have numerator and denominator) have attracted considerable research and interest, since they are useful in production planning, financial and corporate planning, health care and hospital planning. Several methods for solving this problem are proposed in (1962), Charnes and Kooper have proposed their method depends on transforming this (LFP) to an derived from Bitran and Novaes (1973) is used to solve this linear fractional program by solving a sequence of linear programs only re-computing the local gradient of the objective function. Also some aspects concerning duality and sensitivity analysis in linear fraction program was early discussed by Bitran and Magnanti (1976) and Singh in his paper made a useful study about the optimality condition in fractional programming. More recent works on fractional programming theory and methods can be found in[ 5,10,11 ] In this paper we introduce a new concept of duality of a given linear fractional program and this dual is in the form of a linear program .Our new concept is different from the one proposed by [4,6,7,9] where each one of them have defined their duality keeping the primal and dual as linear fractional programs. In Section 2 some notations and definitions of the (LFP) problem is given , while in Section 3 we give the main result together with a simple example to illustrate our results and finally Section 4 gives a conclusion remarks about this proposed duality

## II. DEFINITIONS AND NOTATION

The problem of concern arises when a linear fractional function is to be maximized on a convex polyhedral set X. This problem can be formulated mathematically as follows, and will be denoted by (LFP)

$$\text{Maximize} \quad F(x)=\frac{c^T x+\alpha}{d^T x+\beta}$$

Subject to;

$$x \in X \qquad (1)$$

where $X=\{ x \in R^n, Ax \le b, x \ge 0\}$

Here A is $m \times n$ matrix , c and d are n-vectors, $b \in R^m$ and $\alpha, \beta$ are scalars. . It is assumed that the feasible solution set $X$ is a compact set i. e. bounded and closed. Moreover, $d^T x+\beta \succ 0$ everywhere in $X$.

Also if we assume that $\beta \neq 0$, then an equivalent form of (1) can be formulated as Maximize F(x) =

$$(c^T - \frac{\alpha}{\beta} d^T)\frac{x}{d^T x+\beta} + \frac{\alpha}{\beta} \qquad \text{Subject to}$$

$$(A+\frac{b}{\beta} d^T)\frac{x}{d^T x+\beta} \le \frac{b}{\beta}$$

(2)

$$\frac{x}{d^T x+\beta} \ge 0$$

If we define $y = \dfrac{x}{d^T x+\beta} \ge 0$, then (2) can be written as

Maximize F(y)

$$= \quad (c^T - \frac{\alpha}{\beta} d^T)y + \frac{\alpha}{\beta}$$

Subject to

$$(A+\frac{b}{\beta} d^T)y \le \frac{b}{\beta} \qquad (3)$$

$$y \ge 0$$

The above linear programming can be formulated as

$$\text{Maximize } F(y) = p^T y + \frac{\alpha}{\beta}$$

Subject to

$$Gy \le g \qquad (4)$$

$$y \ge 0$$

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ISO 9001:2008 Certified
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where $p^T = (c^T - \frac{\alpha}{\beta} d^T)$,

$G = (A + \frac{b}{\beta} d^T)$ , and $g = \frac{b}{\beta}$

In the above definition of y, we can get

$$x = \beta \frac{y}{1 - d^T y}$$

This can easily derive as follows , since $y = \frac{x}{d^T x + \beta}$

(5) then $x = y(d^T x + \beta)$ (6)

Operating on both sides by $d^T$ we get $d^T x = d^T y \, (d^T x + \beta)$

Hence $(d^T x + \beta) - \beta = d^T y \, (d^T x + \beta)$ which gives

$(d^T x + \beta) = \beta \frac{1}{1 - d^T y}$, substituting in (6) we get

(5)

**Remark:**

The above linear programming problem (4) is an equivalent transformation for the linear fractional programming problem (1) and has exactly the same dimension and is different from the well known transformation used by [3] where the dimension of the equivalent linear programming problem is increased by one Consider the dual of the linear program of (4) in the form

Minimize $H(u) = u^T g$

Subject to

$u \in U$ (7)

where $U = \{ u \in R^m, u^T G \geq p^T, u \geq 0 \}$

Based on the above definitions for the primal problem (4) and its corresponding dual problem (7) we have the following propositions

**Proposition 2.1: (Weak** duality)

If x is any feasible solution for the primal problem (4) and u is a feasible solution for the dual problem (7), then $F(y) \leq H(u)$

**Proof:** straight forward

**Proposition 2.2** (strong duality)

If y * solves the primal problem (4), then there exists u* which solves the dual problem (7), such that $F(y^*) = H(u^*)$

**Proof:** straight forward

**Proposition 2.3**

(the complementary slackness)

Suppose that s and v are slack and surplus column vectors associated with the primal problem (4), and dual problem given by (7) respectively, then (y* , s* ) solves the primal problem and(u* , v*) solves the dual problem if and only if

$$v^* y^* + s^* u^* = 0$$

Proof: the proof is straight forward since if (x* , s* ) solves the primal problem and (u* , v*) solves the dual problem then by direct duality theorem of linear program we get the above result

Example: Consider the linear fractional programming problem in the form

$$\text{Maximize} \quad z = F(x) = \frac{x_1 + x_2 + 2}{x_1 + 1}$$

Subject to:

$x_1 + x_2 \leq 4$

$-x_1 + 2x_2 \leq 2$

$x_1 \geq 0, x_2 \geq 0$

For this problem we have $c^T = (1 \ \ 1)$, $d^T = (1 \ \ 0)$, also $\alpha = 2$, and $\beta = 1$

Our result shows that the linear programming problem in the form

Maximize $z = F(y) = -y_1 + y_2 + 2$

Subject to:

$5 y_1 + y_2 \leq 4$

$y_1 + 2y_2 \leq 2$

$y_1 \geq 0, y_2 \geq 0$

can be considered as an equivalent linear programming problem which has exactly the same dimension as the original linear fractional programming

Then the dual problem of the above linear programming will be in the form

Minimize $w = F(u) = 4u_1 + 2u_2$

Subject to:

$5 u_1 + u_2 \geq -1$

$u_1 + 2u_2 \geq 1$

$u_1 \geq 0, u_2 \geq 0$

## III. CONCLUSION

In this paper a new concept of duality for single linear fractional programming is introduced .Our new concept is mainly based on transforming the given linear fractional programming problem to an equivalent linear programming problem having exactly the same dimension which is different from the well known transformation used by Charnes and Kooper where the

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ISO 9001:2008 Certified
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Volume 3, Issue 7, January 2014

dimension is increased by one. Also we can extend this duality concept to the case for a special class of multiple objective linear fractional programming problems with the same denominators.

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